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Evidence for large N phase transition in $\mathcal{N} = 4$ super Yang-Mills theory at finite temperature

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Abstract: The AdS/CFT correspondence provides valuable constraints on the possible exact form of various physical quantities in the $\mathcal{N} = 4$ super Yang-Mills theory in the large N limit [1–4]. We examine the free energy as the expansions in a small as well as in a large 't Hooft parameter λ. We argue that it is impossible to smoothly extrapolate from the weak coupling regime to the strong coupling regime, thus there must exist a large N phase transition in λ at a finite temperature. We also argue that there is no world-sheet instanton in the background of the Euclidean anti-de Sitter black hole.

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This is the shortened version of paper `hep-th/9807196`. We will improve the argument for a possible large N strong/weak phase transition in the thermodynamics of $\mathcal{N}=4$ four dimensional super Yang-Mills theory. We will try to make the argument more concise and comprehensive.

In a perturbative calculation of the free energy, the growth of the number of Feynman diagrams should be the same as in calculating a zero temperature physical quantity. The only modification to be made is to replace the usual propagators by the thermal propagators. This modification makes the perturbative behavior of a finite temperature quantity drastically different from its zero temperature counterpart. The major cause is that an electric mass as well as a mass for scalars are generated at one-loop level. This mass scale is proportional to $\sqrt{\lambda}T$. Thus, the perturbative expansion of a physical quantity is no longer analytic in λ, but in $\sqrt{\lambda}$. For instance, the free energy, beyond the two loop result proportional to λ, will have a term $\lambda^{3/2}$ due to the summation of the so-called ring diagrams [1]. Each ring diagram of a bosonic field is infrared divergent, but the summation is finite. The nonanalytic terms generated this way can not be cancelled by exploiting supersymmetry, since there is no such infrared divergent diagram for a fermionic field, due to the fact that a mass for a fermion is already generated at the tree level due to the anti-periodic boundary condition. Thus, we expect the perturbation series of the free energy take the following schematic form

$$F=\sum_{n=0}^{\infty}\lambda^n+\sum_{n=0}^{\infty}\lambda^{\frac{3}{2}+n}, \tag{1}$$

powers of larger half integers can be generated, for instance, by higher order terms in the electric mass. Note that we have assumed terms such as $\lambda^2\ln\lambda$ occurring in QCD are absent here. Such terms, if exist, can be generated in our scheme presented in the next section.

The free energy of the large N SYM was calculated in [2] in the $\lambda=0$ limit, and found to be

$$F=-\frac{\pi^2}{6}N^2V_3T^4\,,$$

where V_3 is the three volume. However the black hole thermodynamics predicts

$$F=-\frac{\pi^2}{8}N^2V_3T^4\,.$$

The mismatch is a factor 3/4. The black hole result is the result for $\lambda=\infty$, thus there is no contradiction here. If the exact formula is $F=-\pi^2/6N^2V_3T^4F(\lambda)$, then $F(0)=1$ and $F(\infty)=3/4$. A two-loop result is not known yet. On general grounds, one expects to have

$$F_{two-loop}=a\frac{\lambda}{\pi^2}\,, \tag{2}$$

where a is a rational number.

Using the stringy correction to the effective action [3, 4], it was found that in the large λ limit the next to leading order correction is [5]

$$\frac{45}{32}\zeta(3)(2\lambda)^{-3/2}\,. \tag{3}$$

It has a positive sign. We rewrite, for reason becomes obvious shortly,

$$F(\lambda) = 3/4 + G(\lambda)\,, \tag{4}$$

then $G(\lambda)$ behaves as $\lambda^{-3/2}$ in the large λ limit, and its value at $\lambda = 0$ is $1/4$.

One of the ways to see the origin the coefficient $\zeta(3)$ of the term $\lambda^{-3/2}$ is through an inspection of the Virasoro-Shapiro amplitude [4]

$$A = \frac{1}{stu}\frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)}\,. \tag{5}$$

Expanding the above formula in α' is equivalent to expanding it in terms of the Mandelstam variables. This gets translated into the effective action as expansion with number of derivatives. At the order $(\alpha')^3$, the coefficient $d^3 \ln\Gamma(x=1)/dx^3$ appears, which is the origin of the factor $\zeta(3)$. At the order $(\alpha')^{n+3}$, many such derivatives will appear, and the highest one is just $d^{n+3}\ln\Gamma(x=1)/dx^{n+3} = (-1)^{n+3}\Gamma(n+3)\zeta(n+3)$. Thus, we expect that the coefficient of the term $\lambda^{-n/2-3/2}$ which corresponds to the order $(\alpha')^{n+3}$ take the form

$$\zeta(n+3) + \zeta(3)\zeta(n) + \cdots\,, \tag{6}$$

where the general term is a product of zeta's whose arguments sum to $n+3$, and each argument is an odd integer. Thus, we expect the coefficient of $\lambda^{-n/2}$ to contain a term $\zeta(n)$.

In the following argument, we will not make any assumption about the singular behavior of $G(\lambda)$ on the complex plane. Our aim is to find plausible contradiction with the assumption that $G(\lambda)$ is smooth on the positive axis. However, one physical assumption about the singular behavior at the end of $\lambda = \infty$ is needed:

1. The point $\lambda = \infty$ is not an essential singularity of the function $F(\lambda)$, or the function $G(\lambda)$.

We will argue for this assumption in the end of this article. Another point, which is not needed for our argument, is the following.

2. $\lambda = 0$ is not an essential singularity of $F(\lambda)$ too.

This is only true for the $\mathcal{N} = 4$ theory under consideration. For a finite N, $g^2 = 0$ is an essential singularity due to the existence of Yang-Mills instantons. As argued in [6], such effects vanish in the large N limit.

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Assuming that $G(\lambda)$ is a smooth function on the whole positive axis, then the following Mellin transform

$$G^*(s) = \int_0^\infty G(\lambda)\lambda^{s-1} d\lambda \tag{7}$$

exists for $0 < \Re s < 3/2$, owing to the decay behavior $G(\lambda) \sim \lambda^{-3/2}$ for large λ. Note that the function $F(\lambda)$ does not have a Mellin transform. Given $G^*(s)$ defined for a range of s, it can be analytically extended to the whole complex plane of s. Since $G(\lambda)$ is assumed to be a smooth function on the positive axis, $G^*(s)$ is analytic everywhere except at poles.

$G(\lambda)$ is determined by $G^*(s)$ through the inverse Mellin transform

$$G(\lambda) = \frac{1}{2\pi i}\int_C ds(2\lambda)^{-s} G^*(s)\,, \tag{8}$$

where the contour C is a straight line with a constant $\Re s$ satisfying $0 < \Re s < 3/2$, we also rescaled $G^*(s)$ by a factor 2^{-s}. According to the discussion in the last section, $G^*(s)$ must have poles at all integers and half integers with three exceptions $s = \pm 1/2$, $s = 1$. For a small λ, the contour is closed on the left half plane, picking up poles at negative integers and negative half integers. Since the small λ series is convergent for a sufficient small λ, thus the function $G^*(s)$ must grow at most as an exponential function for large negative $\Re s$. On the other hand, there is no such constraint for a large positive $\Re s$. Thus, for a large enough λ, the contour can not be closed on the right half plane. However, one can shift the contour rightward, thus gradually picks up negative powers in λ, and eventually the expansion could be only an asymptotic expansion.

In fact, for $G^*(s)$ to exist, $G(\lambda)$ does not have to be smooth. For instance, it can be a step function. An example is $G(\lambda) = \theta(1-\lambda)$. In this case $G^*(s) = 1/s$. It has only one singularity. However, our $G(\lambda)$ has both a small λ expansion as well as a large λ expansion, so $G^*(s)$ has poles on the whole real axis. To rule out functions such as step functions and piece-wise smooth functions, it is important to insist on that poles on the left of contour C are related to poles on the right of C. For a $G(\lambda)$ whose Mellin transform $G^*(s)$ has only poles on the left, it is either a step function, or a function with an essential singularity at $\lambda = \infty$. We will make use of this argument explicitly.

Thus one can write

$$G^*(s) = \frac{(s^2 - 1/4)(s-1)}{\sin(2\pi s)}\left(\frac{\Gamma(a)\zeta(2s)}{\Gamma(a-2s)}f(s) + \cdots\right), \tag{9}$$

where $f(s)$ is not specified yet. The above ansatz incorporates both the known small λ and the large λ expansion. For the inverse Mellin transform to exist, $f(s)$ can not grow faster than an exponential for large $|s|$. $\zeta(2s)$ is included to reproduces the coefficients in (6). It vanishes if s is a negative integer. The ansatz still has poles on the both sides on the contour C. This is to avoid the problem mentioned in the last paragraph. For the integral (8) with the integrand given in (9) to make sense, $|f(s)|$ can grow no faster than the exponential $\exp(\pi|\Im s|)$ for large $|\Im s|$. Also, in order to close the contour on the left half plane to reproduce the perturbation series, $|f(s)|$ should not grow faster than an exponential function for a large negative $\Re s$.

Now the inverse Mellin transform dictates an asymptotic expansion for a large λ:

$$G(\lambda) = -\frac{\sin(\pi a)\Gamma(a)}{16\pi^2}\sum_{n=0}^{\infty}(n+1)(n+2)(n+4)\Gamma(n+4-a)f\left(\frac{n}{2}+\frac{3}{2}\right)\times \\ \times\,\zeta(n+3)(2\lambda)^{-n/2-3/2}+\cdots \tag{10}$$

where we have the desired factor $\zeta(n+3)$. For $n=0$, the other factors should comprise to give 45/32.

$\zeta(2s)$ vanishes if s is a negative integer. It has a simple pole at $s=1/2$. This is a undesired feature, thus $f(1/2)=0$ in order not to have a term $\lambda^{-1/2}$ in the large λ expansion. Although $\zeta(2s)$ does not give rise to positive integral powers, it gives rise to positive half-integral powers as well as to the constant term

$$G(\lambda) = -\frac{1}{16\pi}f(0)+\sum_{n=1}^{\infty}(-1)^{n+1}(n^2+n)(2n+3)\frac{\zeta(2n+2)f(-n-\frac{1}{2})}{4\pi^2}\times \\ \times\frac{\Gamma(a)\Gamma(2n+2)}{\Gamma(2n+1+a)}\left(\frac{\lambda}{2\pi^2}\right)^{n+1/2}. \tag{11}$$

This series is convergent, since $f(-n-1/2)$ by definition does not grow faster than an exponential function. The general perturbation theory indicates that all the π factor is what already included in $(\lambda/2\pi^2)^{n+1/2}$, and the zeta function $\zeta(2n+2)$ is π^{2n+2} multiplied by a rational number, thus $f(-n-1/2)\sim\pi^{-2n}$. If so, since for large n, $\zeta(2n+2)\to 1$, the convergent radius does not contain a factor π^2. Indeed, if $f(-n-1/2)$ has the same sign for sufficiently large n, the singular point λ_c will be negative. Notice that the constant term must be a rational number (1/4), this requires $f(0)\sim\pi$, and this is consistent with the pattern $f(-n-1/2)\sim\pi^{-2n}$.

However, the coefficients in (10) must be rational, except for the factor $\zeta(n+3)$. It is 45/32 at $n=0$ as computed in [5]. It is fairly easy to see from the strategy of that calculation that one will get a rational number at the order $(\alpha')^{n+3}$, for each factor $\zeta(n+3)$, $\mathbf{C}\zeta(n+2)$ etc. Now the factor $\sin(\pi a)\Gamma(a)\Gamma(n+4-a)$ in (10) is a rational number times π. Thus, $f((n+3)/2)$ must be a rational number times π to cancel the factor π^2 in the denominator in (10). The only way to solve this problem is to use

$$f(s)=\pi^{2s+1}h_1(s)+\pi h_2(s)\,, \tag{12}$$

such that $h_1(s)$ has zeros at $s = (n+3)/2$, while $h_2(s)$ has zeros at $s = -(n+3/2)$. Both $h_i(s)$ must be bounded by exponential functions for a general s. It is quite easy to construct functions with the these properties.

It appears therefore that it is possible to interpolate between the small λ expansion and the large λ expansion. However, once (12) is used in the contour integral (8), we see that $h_1(s)$ does not contribute to the large λ expansion, therefore gives rise to an essential singularity at $\lambda = \infty$. This contradicts our assumption 1.

We have assumed terms such as $\lambda^2 \ln \lambda$ are absent. These terms can be included into our consideration by introducing double poles in $G^*(s)$.

Finally, we have to justify the assumption that $\lambda = \infty$ is not an essential singularity. Assume the opposite, this would imply that the world-sheet theory in the black hole background has an essential singularity in the α' expansion, since this is equivalent to the expansion in terms of $1/\sqrt{\lambda}$. Here we provide two related arguments that this is not the case.

The first argument has to do with the world-sheet instantons. That the α' expansion has an essential singularity at the origin is a familiar phenomenon. This occurs for instance in the case of the Calabi-Yau compactifications. In all known examples, there are world-sheet instantons corresponding to the string world-sheet wrapped around a two-cycle. There is no topologically nontrivial two-cycles in the background of the Euclidean AdS black hole. The next possible candidate is the spherical world sheet wrapped doubly around the tip of the black hole. There is no minimal surface of this type. The reason is that the two dimensional metric in the subspace spanned by the radial coordinate and the Euclidean time coordinate is $ds^2 \sim d\rho^2 + \sinh\rho \tanh\rho d\psi^2$. The radius of ψ increases monotonically with ρ. This argument suggests that the modified geometry, determined by the exact CFT, will be regular series in the α' expansion.

Our second argument has to do with a four point correlation function. If an essential singularity shows up in the free energy, it is likely to show up in a correlation function. One can calculate the four point function of the stress-tensor operator using the four point string amplitude in the black hole background. The four point amplitude as given in (5) has no essential singularity at $\alpha' = 0$. The only possible origin is then the modified geometry of the black hole due to α' corrections. As discussed above, the geometry does not have an essential singularity either.

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